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Refined similarity hypotheses for passive scalars mixed by turbulence

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In analogy with Kolmogorov's refined similarity hypotheses for the velocity field, two hypotheses are stated for passive scalar fields mixed by high-Reynolds-number turbulence. A 'refined' Yaglom equation is derived under the new assumption of local isotropy in pure ensembles, which is stronger than the usual assumption of local isotropy but weaker than the isotropy of the large scale. The new theoretical result is shown to be consistent with the hypotheses of refined similarity for passive scalars. These hypotheses are approximately verified by experimental data on temperature fluctuations obtained (in air) at moderate Reynolds numbers in the wake of a heated cylinder. The fact that the refined similarity hypotheses are stated for high Reynolds (and Péclet) numbers, but verified at moderate Reynolds (and Péclet) numbers suggests that these hypotheses are not sufficiently sensitive tests of universality. It is conjectured that possible departures from universality are hidden by the process of taking conditional expectations.

1. Introduction

A substantial part of our understanding of high-Reynolds-number turbulence comes from the scaling theory proposed by Kolmogorov (1941*a*), relating the statistics of velocity increments to the average dissipation rate of kinetic energy. The necessity to account for strong fluctuations in energy dissipation rate (Batchelor & Townsend 1948; Landau & Lifshitz 1959), not taken into account in Kolmogorov (1941*a*), led Obukhov (1962) and Kolmogorov (1962) to reformulate the original theory. The refinements, stated succinctly by Kolmogorov (1962), have become known as refined similarity hypotheses. Even though some consequences of Kolmogorov's refined theory have been indirectly verified experimentally in the past (e.g. Anselmet *et al.* 1984; Meneveau & Sreenivasan 1991), it is only recently that the basic tenets underlying the refined hypotheses have been verified in some detail (Stolovitzky, Kailasnath & Sreenivasan 1992; Praskovsky 1992; Thoroddsen & Van Atta 1992*a*, Chen *et al.* 1993; Hosokawa 1993). A theoretical approach to this problem was taken by Stolovitzky & Sreenivasan (1994).

The scaling theory of fluctuations of a passive scalar field has been developed by Obukhov (1949), Yaglom (1949), Corrsin (1951), Batchelor (1959) and Batchelor, Howells & Townsend (1959), along lines that are similar in various degrees to that for the velocity field. Hints about the modifications needed to account for strong fluctuations in the local rates of dissipation for both the velocity and scalar fields can be found dispersed in Monin & Yaglom (1975), and some consequences of these modifications have been discussed in various forms, for example, by Van Atta (1971), Antonia & Van Atta (1975), Meneveau *et al.* (1990) and Sreenivasan (1991). Hosakawa (1994) proposed an extension of the refined hypothesis for the passive scalar and used this extension to compute the probability density function of temperature increments and gradients. His results compared reasonably well with the temperature increment data of Antonia *et al.* (1984) and with the temperature gradient data of Thoroddsen & Van Atta (1992*b*). This work therefore constitutes, albeit indirectly, a confirmation of the refined similarity hypotheses for the passive scalar. However, the refined hypotheses for the passive scalar have not yet been stated in their most general form, and have not been subjected to direct experimental scrutiny. In this paper we state them in a general way and present some experimental support; we further obtain specific results from the dynamical equation governing the passive scalar, and show that predictions of the hypotheses are consistent with them.

After providing the needed background and a description of notation in §2, we state in §3 Kolmogorov's refined similarity hypotheses for the velocity field (henceforth denoted by KRSH) as well as the analogous refined similarity hypotheses for the passive scalar field (henceforth denoted by RSHP). The principal theoretical result deduced from the evolution equation for the passive scalar is described in §4, and the sense in which this result supports RSHP is pointed out. Experimental details and results are described in §§5 and 6 respectively. The paper concludes with a few summary remarks in §7.

2. Background and notation

Let $\theta(\boldsymbol{x},t)$ denote a scalar field θ at position $\boldsymbol{x}$ and at time t. Imagine that this scalar is mixed by the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ of the fluid in turbulent motion. Let ν and χ denote, respectively, the kinematic viscosity and kinematic diffusivity of the fluid. We shall think of θ as the temperature. Needless to say, other scalars such as the concentration of a dye are amenable to the same analysis. The Prandtl number $Pr = \nu/\chi$. Two quantities that play main roles in this study are the energy dissipation rate per unit mass

$$\epsilon(\boldsymbol{x},t) = \frac{\nu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2 \tag{2.1}$$

and the scalar dissipation rate per unit mass

$$N(\boldsymbol{x},t) = \chi\sum_{i=1}^{3}\left(\frac{\partial \theta}{\partial x_i}\right)^2 . \tag{2.2}$$

Under the assumption of local isotropy (to which we shall return shortly), it is well-known that $\langle\epsilon\rangle = 15\nu\langle\,(\partial u_1/\partial x_1)^2\,\rangle$ and $\langle N\rangle = 3\chi\langle\,(\partial\theta/\partial x_1)^2\,\rangle$, where $\langle\ldots\rangle$ denotes an ensemble average. Relations (1) and (2) have been used extensively in the literature; also used extensively (sometimes under questionable circumstances) as surrogates of the local rates of energy and temperature dissipation are $\epsilon' = 15\nu\,(\partial u_1/\partial x_1)^2$ and $N' = 3\chi\,(\partial\theta/\partial x_1)^2$. Clearly, if local isotropy prevails, $\langle\epsilon'\rangle = \langle\epsilon\rangle$ and $\langle N'\rangle = \langle N\rangle$. Because ϵ' and N' are both highly intermittent quantities (Batchelor & Townsend 1948; Sreenivasan & Meneveau 1988; Meneveau *et al.* 1990) the small structure of the turbulence field is better related to local averages of the two dissipation rates rather than to their global averages. Denote by $N_r(\boldsymbol{x}_0,t)$, the local average of the scalar dissipation rate over a region of linear size r around the location $\boldsymbol{x}_0$,

$$N_r(\boldsymbol{x}_0,t) = \frac{6}{\pi r^3}\int_{B(\boldsymbol{x}_0,r)} \mathrm{d}^3\xi N(\xi,t) \tag{2.3}$$

where $B(\boldsymbol{x}_0,r/2)$ is a sphere of radius $r/2$ centred at $\boldsymbol{x}_0$. Alternatively, the local average in the case of the one-dimensional surrogate of N will be

$$N_r'(x_0,t) = \frac{1}{r}\int_{x_0}^{x_0+r} \mathrm{d}\xi N'(\xi,t). \tag{2.4}$$

Equivalent definitions for the local averages of the energy dissipation rate can be obtained by replacing N by ϵ in equations (2.3) and (2.4). Obukhov (1962) argued that the small structure of turbulence is related to averages performed over the *pure* ensemble: by the pure ensemble at a fixed r and $\boldsymbol{x}_0$, Obukhov meant the set of realizations of turbulent velocity and scalar fields for which the dissipation of temperature and energy take given values $N_r(\boldsymbol{x}_0,t) = n$ and $\epsilon_r(\boldsymbol{x}_0,t) = e$.

To describe features of turbulent flows that have any generality to them, we must restrict attention to scales r which are much smaller than a typical external scale L – this being the smaller of the integral scales for the velocity and scalar fields. The local structure of the flow is best described by quantities such as the instantaneous difference of the temperature and velocity between two points in space $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$; that is, in terms of temperature difference $\Delta\theta(\boldsymbol{x},\boldsymbol{r}) = \theta(\boldsymbol{x}+\boldsymbol{r}) - \theta(\boldsymbol{x})$ and the velocity difference $\Delta\boldsymbol{u}(\boldsymbol{x},\boldsymbol{r}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$. For notational simplicity we shall at times write $\Delta_r\theta$ instead of $\Delta\theta(\boldsymbol{x},\boldsymbol{r})$ and $\Delta_r\boldsymbol{u}$ instead of $\Delta\mathrm{u}(\mathrm{x},\boldsymbol{r})$. We shall denote by $\Delta_r u_r$ the scalar field produced by projecting $\Delta\boldsymbol{u}(\boldsymbol{x},\boldsymbol{r})$ onto the direction of $\boldsymbol{r}$, i.e., $\Delta_r u_r = \boldsymbol{r}/r\cdot\Delta\boldsymbol{u}(\boldsymbol{x},\boldsymbol{r})$. With this as the background, the refined similarity hypotheses can now be stated.

3. The refined similarity hypotheses

3.1. *Kolmogorov's refined similarity hypotheses for the velocity field (KRSH)*

Even though the KRSH are not the main point of this paper, it will later prove useful to state them here. In the form used here, KRSH for fully developed turbulence can be stated as follows:

The first hypothesis: For a range of scales r such that $r \ll L$, the conditional probability density function (p.d.f.) of the stochastic variable

$$V = \frac{\Delta u_r(\boldsymbol{x},\boldsymbol{r})}{(r\epsilon_r)^{1/3}} \tag{3.1}$$

(where $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$ are poles of the sphere $B(\boldsymbol{x}_0,r/2)$) given that $\epsilon_r(\boldsymbol{x}_0) = \epsilon_r$, is independent of $\boldsymbol{x}$ and $\boldsymbol{x}_0$ and depends only on the local Reynolds number $Re_r = r(r\epsilon_r)^{1/3}/\nu$.

The second hypothesis: For $Re_r \gg 1$ (or, equivalently, for $r \gg \eta_r$, where $\eta_r = (\nu^3/\epsilon_r)^{1/4}$ is the local Kolmogorov thickness), the conditional p.d.f. of V for a given ϵ_r becomes independent of Re_r, and is therefore universal.

There is also a third hypothesis concerning the lognormality of ϵ_r about which we shall have nothing to say in this paper.

It turns out that these hypotheses, when stated in terms of the one-dimensional surrogate ϵ', have to be modified to allow for an additional r-dependence of the p.d.f. of V. The reason is the following. Consider a conditional ensemble of realizations of turbulence with a fixed $\epsilon_r'(x_0) = \epsilon_r'$. If the turbulence is assumed locally isotropic, the direction $\boldsymbol{r}/r$ of the one-dimensional cut along which ϵ'_r is computed is irrelevant. This ensemble has a non-empty intersection with the ensemble of fixed $\epsilon_r(\mathbf{x}_0) = \delta$, for δ between a minimum and maximum value (which may well be zero and infinity). In the ensemble of fixed ϵ'_r, the statistics of $V' = \Delta u_r/(r\epsilon'_r)^{1/3}$ can be computed as an appropriate average of the statistics of $V = \Delta u_r/(r\epsilon_r)^{1/3}$, weighted by the *r-dependent* relative contribution of ϵ_r to the ensemble of fixed ϵ'_r†. Therefore, the statistics of V' will inherit its local Reynolds number dependence (now in terms of ϵ'_r) from the statistics of V, and its r-dependence from the conditional statistics of ϵ_r given ϵ'_r.

In Stolovitzky & Sreenivasan (1994) and in Stolovitzky (1994) it was indeed shown that the ϵ'_r-based local Reynolds number Re'_r does not uniquely determine the conditional p.d.f. of V', and that an additional dependence on r/η (where $\eta = (\nu^3/\langle\epsilon\rangle)^{1/4}$ is the Kolmogorov thickness) enters the picture. Therefore, the appropriate statements of the refined similarity hypotheses for the velocity field when expressed in terms of ϵ'_r are the following:

The first hypothesis (in terms of ϵ'_r): For a range of scales r such that $r \ll L$, the p.d.f. of the stochastic variable

$$V' = \frac{\Delta u_r(\boldsymbol{x}, \boldsymbol{r})}{(r\epsilon'_r)^{1/3}} \tag{3.2}$$

conditioned on ϵ'_r is independent of $\boldsymbol{x}$ and depends only on r/η and the local Reynolds number $Re'_r = r(r\epsilon'^2_r)^{1/3}/\nu$.

The second hypothesis (in terms of ϵ'_r): For $r \gg \eta$ and $Re_r \gg 1$, the conditional p.d.f. of V' given ϵ'_r becomes independent of r and Re'_r, and is therefore universal.

One of the implications of the second hypothesis is that a correlation exists between $|\Delta u_r|$ and ϵ_r. As an experimental fact, this correlation depends on precisely which component of ϵ_r is used as its surrogate. Thoroddsen (1995) pointed out that the use of the surrogate $\epsilon^* \sim (\partial w/\partial x)^2$, where w is velocity component transverse to the x-direction, reduces the correlation from that obtained using ϵ'_r. His work stresses that the choice of ϵ'_r in the previous work was somewhat fortuitious, besides having been simply convenient. Thoroddsen's point has been confirmed by Chen *et al.* (1995) in their numerical simulations and also in our own high-Reynolds-number experiments in the atmospheric surface layer (to be published). It should be stressed, however, that a demonstrable correlation between $|\Delta u_r|$ and ϵ_r persists even if the full definition of the dissipation, or any of its various other surrogates, is used (Chen *et al.* 1995); in particular, this correlation remains at a non-trivial level in high-Reynolds-number atmospheric turbulence even when Thoroddsen's surrogate is used.

3.2. *The refined similarity hypotheses for passive scalars (RSHP)*

The analogous hypotheses for a passive scalar mixed by fully developed turbulence can be stated now as follows:

The first hypothesis: For a range of r such that $r \ll L$, the conditional p.d.f. of the stochastic variable

$$V_\theta = \Delta\theta(\boldsymbol{x}, \boldsymbol{r})\frac{(r\epsilon_r)^{1/6}}{(rN_r)^{1/2}} \tag{3.3}$$

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† For fixed r, this relative contribution is ruled by the conditional probability $P(\epsilon_r|\epsilon'_r;r)$ of ϵ_r given ϵ'_r. That this conditional probability depends on r can be seen by studying its behaviour for large and small r. For $r \rightarrow L$ the probability that ϵ_r be different from ϵ'_r tends to zero, while for very small r the same probability is different from zero. In effect, for very small r, one has that $\epsilon_r \approx \epsilon'_r + X$, where $X = \frac{1}{2}\nu\sum(\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2 - \frac{14}{15}\epsilon'_r$, where the sum is over all the indices $i = 1,2,3$ and $j = 1,2,3$ excluding the case of i and j being simultaneously 1. The probability that ϵ'_r differs from ϵ_r (i.e. the probability that X be different from zero) is clearly positive.

(where x and $x+r$ are poles of the sphere $B(x_0, r/2)$) for given $\epsilon_r(x_0) = \epsilon_r$ and $N_r(x_0) = N_r$, is independent of x and x_0, and depends only on the local Reynolds number $Re_r = r(r\epsilon_r)^{1/3}/\nu$ and the local Péclet number $Pe_r = Re_r Pr$.

The second hypothesis: For $Re_r \gg 1$ and $Pe_r \gg 1$ (or what is the same, for $r \gg \eta_{max}(\epsilon_r)$, where $\eta_{max}(\epsilon_r) = \max\{(\nu^3/\epsilon_r)^{1/4}, (\chi^3/\epsilon_r)^{1/4})\}$, the conditional p.d.f of V_θ for given ϵ_r and N_r becomes independent of Re_r and Pe_r, and is therefore universal.

As already remarked, these hypotheses for the special case of unity Prandtl number have been advanced by Hosakawa (1994), who assumed that V_θ was independent of r, ϵ_r, and N_r as a first approximation.

For reasons already discussed in the context of KRSH, it is reasonable to postulate an additional r-dependence of the refined similarity hypotheses for passive scalars when stated in terms of the one-dimensional surrogates of the dissipation of the energy and the scalar.

The first hypothesis (in terms of N'_r and ϵ'_r): For a range of r such that $r \ll L$, the p.d.f. of the stochastic variable

$$V'_\theta = \Delta\theta(x,r)\frac{(r\epsilon'_r)^{1/6}}{(rN'_r)^{1/2}} \tag{3.4}$$

conditioned on ϵ'_r and N'_r is independent of x and depends only on $r/\eta_{max}(\langle\epsilon\rangle)$ (where $\eta_{max}(\langle\epsilon\rangle) = \max\{(\chi^3/\langle\epsilon\rangle)^{1/4}, (\nu^3/\langle\epsilon\rangle)^{1/4}\}$, the local Reynolds number $Re'_r = r(r\epsilon'_r)^{1/3}/\nu$ and the local Péclet number $Pe'_r = Re'_r Pr$.

The second hypothesis (in terms of N'_r and ϵ'_r): For $r \gg \eta_{max}(\langle\epsilon\rangle)$, $Re'_r \gg 1$ and $Pe'_r \gg 1$, the conditional p.d.f. of V'_θ given ϵ'_r and N'_r becomes independent of r, Re'_r and Pe'_r, and is therefore universal.

3.3. *A remark on the r-dependence in the refined hypotheses*

The standard practice in experiments, necessitated by convenience, is to use one term to represent the total dissipation. This calls for the introduction of some modifications in the context of KRSH. This suggests that experimental tests of the hypotheses (in which ϵ' is usually considered) can be usefully complemented by the direct numerical simulations (DNS) of turbulence, in which both ϵ' and the full energy dissipation rate ϵ can be obtained. Such studies have been made at moderate Reynolds numbers by Chen *et al.* (1993) and Wang *et al.* (1994). These latter authors have shown that the r-dependence introduced by considering KRSH in terms of ϵ'_r tends to disappear when the full ϵ_r is considered as the conditioning parameter. One might expect that the effect of using N' instead of N in RSHP should be more benign than the equivalent of considering ϵ' instead of ϵ in KRSH: as argued in Sreenivasan, Antonia & Danh (1977), the approximation of using the one-dimensional surrogate is in general better for the scalar dissipation rate.

Our goal in the remainder of this paper is to provide support to RSHP. In the following section we will show that RSHP are consistent with results derived directly from the equations of motion under some reasonable assumptions. In §6, we will test RSHP as stated in terms of the one-dimensional surrogates ϵ' and N'. It would be highly desirable that similar tests be performed using DNS of passive scalars mixed by turbulence, considering the full expressions for both dissipation rates.

4. Theoretical analysis

4.1. *The statement of the principal result*

Under the assumptions of local isotropy in each of the conditional ensembles of fixed ϵ_R and N_R, we wish to prove that Yaglom's (1949) equation,

$$\langle \Delta_r u_r (\Delta_r \theta)^2 \rangle = -\tfrac{4}{3} \langle N \rangle r + 2\chi \frac{\partial}{\partial r} \langle (\Delta_r \theta)^2 \rangle, \tag{4.1}$$

can be written in terms of conditional expectations in the pure ensembles. In equation (4.1), $\langle \ldots \rangle$ is the ensemble average in G, where G is a global domain of size equal to the integral scale L in which isotropy obtains. In the pure ensemble of fixed $\epsilon_R = e$ and $N_R = n$, the equivalent equation is

$$\langle \Delta_r u_r (\Delta_r \theta)^2 | e, n; R \rangle = -\tfrac{4}{3} n \, r + 2\chi \frac{\partial}{\partial r} \langle (\Delta_r \theta)^2 | e, n; R \rangle. \tag{4.2}$$

In this equation, the ball of radius $R/2$ defining the pure ensemble contains the points $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{r}$ across which the increments $\Delta_r u_r$ and $\Delta_r \theta$ are taken.

Equation (4.1) was originally derived by Yaglom (1949) under the assumption that the velocity and temperature fields were jointly isotropic. In their derivation of Kolmogorov's (1941*b*) structure equation under the assumption of local isotropy, Monin & Yaglom (1975) pointed out that the same derivation could be used to obtain Yaglom's equation, without requiring the isotropy of the large scale. Later in this subsection we will derive equation (4.2) under the assumption of local isotropy in the pure ensembles, following the spirit of Monin & Yaglom's derivation of Kolmogorov's structure equation. Thus, equation (4.2) holds rigorously if the assumption of local isotropy in the pure ensembles were applicable. This assumption is more restrictive than the usual assumption of local isotropy within a global domain G. In effect, our assumption implies that the locally isotropic ensemble of realizations of turbulence in G has subsets (the pure ensembles), which are locally isotropic in any of the spatial subregions given by the balls $B(\boldsymbol{x}, R/2)$. A direct assessment of the validity of this assumption for turbulence† will have to await further work. If there were departures from local isotropy in pure ensembles, equation (4.2) would stand up only as an approximation. In the remainder of this section we shall assume that the hypothesis of local isotropy in the pure ensembles holds strictly and concentrate on its consequences.

The optimum way of using the information that the local averages of dissipation of energy and scalar in the domain $B(\boldsymbol{x}_0, R/2)$ are e and n when computing the moments of $\Delta_r u_r$ and $\Delta_r \theta$, is to set $r = |\boldsymbol{r}| = R$. Physically this means that the points $\boldsymbol{x} + \boldsymbol{r}$ and $\boldsymbol{x}$ (across which the increment $\Delta_r u_r$ and $\Delta_r \theta$ are taken) are poles of the sphere $B(\boldsymbol{x}_0, R/2)$. It can be proved easily that for $r = R$, the second term on the right-hand

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† The one-dimensional version of this assumption, namely local homogeneity in the pure ensembles, can be shown to hold true for a variety of one-dimensional stochastic processes such as classical Brownian motion (which is a process with independent increments) as well as for fractional Brownian motion (which is a process with correlated increments) when the pure ensembles are defined via a quantity similar to the one-dimensional surrogate of the energy dissipation. The validity of this assumption, however, depends on both the type of the stochastic process and the conditioning defining the pure ensemble. For example, if f is a homogeneous, twice-differentiable Gaussian process, the process given by $\mathrm{d}^2 f / \mathrm{d}x^2$ is not locally homogeneous in the pure ensembles for the scale size $r \to 0$ constrained by a fixed value of f (R. H. Kraichnan 1994, personal communication).

side of equation (4.2) can be written in the form

$$\frac{\partial}{\partial r}\langle(\Delta_r\theta)^2|e,n;R\rangle|_{r=R} = \frac{\partial}{\partial R}\langle(\Delta_R\theta)^2|e,n;R\rangle - \frac{\partial}{\partial\xi}\langle(\Delta_R\theta)^2|e,n;\xi\rangle|_{\xi=R}. \tag{4.3}$$

In interpreting equation (4.3), it is important to note that $\langle(\Delta_r\theta)^2|e,n;R\rangle$ is a function of r, e, n and R, while $\langle(\Delta_R\theta)^2|e,n;R\rangle$ is a function of e, n and R only. In the former expression, r can take any value between 0 and R, and the pure ensemble over which the average is taken, $\epsilon_R = e$ and $N_R = n$, is fixed. In the latter expression, the increments are taken across their saturated value (the diameter of the ball defining the ensemble), and in changing R keeping e and n fixed, we are changing the ensemble. In the second term on the right-hand side of equation (4.3), we are taking a derivative with respect to the diameter ξ of the sphere that determines the pure ensemble, keeping fixed the two points across which the increment $\Delta_R\theta$ is taken, and evaluating the result in $\xi = R$.

Using equation (4.3), we can rewrite equation (4.2) at $r = R$ as

$$\langle\Delta_r u_r(\Delta_r\theta)^2|e,n;r\rangle = -\tfrac{4}{3}nr + 2\chi\frac{\partial}{\partial r}\langle(\Delta_r\theta)^2|e,n;r\rangle - 2\chi\frac{\partial}{\partial\xi}\langle(\Delta_r\theta)^2|e,n;\xi\rangle|_{\xi=r}. \tag{4.4}$$

When the local Reynolds and Péclet numbers are much larger than unity, the right-hand side of equation (4.4) is dominated by the first term. In effect, noting from the second KRSH that for $Re_r \gg 1$, $\Delta_r u_r \sim (r\epsilon_r)^{1/3}$, and that $\langle\Delta_r u_r(\Delta_r\theta)^2|e,n;r\rangle \sim (r\epsilon_r)^{1/3}\langle(\Delta_r\theta)^2|e,n;r\rangle$, it follows that

$$\chi\frac{\partial}{\partial r}\langle(\Delta_r\theta)^2|e,n;r\rangle \sim \chi\frac{\partial}{\partial\xi}\left.\langle(\Delta_r\theta)^2|e,n;\xi\rangle\right)_{\xi=r} \sim \chi\frac{\langle(\Delta_r\theta)^2|e,n;r\rangle}{r} \sim \frac{rn}{Pe_r}.$$

Therefore, for $Re_r \gg 1$ and $Pe_r \gg 1$, we have

$$\langle\Delta_r u_r(\Delta_r\theta)^2|e,n;r >\approx -\tfrac{4}{3}nr. \tag{4.5}$$

Another interesting result can be obtained from equation (4.4) in the limit of $r \ll \eta_{min}$ where $\eta_{min} = \min\{(\chi^3/e)^{1/4}, (\nu^3/e)^{1/4}\}$. In such a case, the left-hand side of equation (4.4) can be neglected (for it is of third order in r which is assumed very small), and the last term on the right-hand side is zero because for $\xi > r$, $\langle(\Delta_r\theta)^2|e,n;\xi\rangle = \langle(\hat{r}\cdot\nabla\theta)^2 r^2|e,n;r\rangle = n/3\chi r^2$ and therefore $(\partial/\partial\xi)\langle(\Delta_r\theta)^2|e,n;\xi\rangle = 0$. It follows that the second term on the right-hand side of equation (4.4) has to cancel the first term. Thus

$$\langle(\Delta_r\theta)^2|e,n;r\rangle \approx \frac{n}{3\chi}r^2. \tag{4.6}$$

The averages over e and n of equations (4.5) and (4.6) yield the known results (Yaglom 1949)

$$\langle\Delta_r u_r(\Delta_r\theta)^2\rangle \approx -\tfrac{4}{3}\langle N\rangle r \tag{4.7}$$

and

$$\langle(\Delta_r\theta)^2\rangle \approx \frac{\langle N\rangle}{3\chi}r^2 \tag{4.8}$$

for r sufficiently large and small respectively.

It is interesting to discuss an important difference between the similarly looking equations (4.5) and (4.7). Physically, Yaglom's relation $\langle\Delta_r u_r(\Delta_r\theta)^2\rangle = -\tfrac{4}{3}\langle N\rangle r$ indicates that the flux of scalar fluctuations $(\Delta_r\theta)^2$ at scale r towards the small scales is controlled by the global mean of scalar dissipation rate $\langle N\rangle$. This is the expression of a global balance of scalar fluctuations: the size of the fluctuations transferred (towards smaller scales) across any given inertial–convective scale is only dissipated at the smallest scales. The relation $\langle \Delta_r u_r (\Delta_r \theta)^2 | e, n; r \rangle = -\frac{4}{3} n r$ expresses the same kind of balance of scalar fluctuations, but in a detailed fashion. By the latter, we mean that the mean flux of scalar fluctuations in the pure ensemble at fixed $N_r = n$ is controlled not by the global average $\langle N \rangle$, but by the local average N_r.

Given that detailed balance implies global balance, we can derive equations (4.5) and (4.6) from equations (4.7) and (4.8). However, as global balance does not imply detailed balance, the validity of equations (4.7) and (4.8) does not, in principle, imply the validity of equations (4.5) and (4.6). These last two relations follow from equation (4.2), which in turn is warranted if the hypothesis of local isotropy in the pure ensembles is postulated. This assumption is stronger than the assumption of local isotropy in the global domain (as was pointed out before) and weaker than the assumption of global isotropy, in that it does not demand the isotropy of the large scales of motion. Therefore, and for the sake of completeness, it seems appropriate to derive equation (4.2) without using the assumption of global isotropy used in the classical derivation of Yaglom's equation, but with the assumption of local isotropy in pure ensembles.

4.2. *The derivation of equation (4.2)*

Let $\boldsymbol{r}'$ and $\boldsymbol{r}''$ be two points within the domain G where the turbulence is locally isotropic. Denote $\boldsymbol{u}' = \boldsymbol{u}(\boldsymbol{r}', t)$, $\boldsymbol{u}'' = \boldsymbol{u}(\boldsymbol{r}'', t)$, $\theta' = \theta(\boldsymbol{r}', t)$ and $\theta'' = \theta(\boldsymbol{r}'', t)$. To derive equation (4.2) without any special assumption about the isotropy of the large-scale structure of the flow, we must transform the advection–diffusion equation into an equation that contains only velocity and temperature differences and their derivatives. We start by writing the advection–diffusion equation at the point $\boldsymbol{r}''$:

$$\frac{\partial \theta''}{\partial t} + u_j'' \frac{\partial \theta''}{\partial r_j''} = \chi \frac{\partial^2 \theta''}{\partial r_j'' \partial r_j''}. \tag{4.9}$$

Next, we add and subtract from the left-hand side of equation (4.9) the term $u_j' \partial \theta'' / \partial r_j''$, and obtain

$$\frac{\partial \theta''}{\partial t} + u_j' \frac{\partial \theta''}{\partial r_j''} + [u_j'' - u_j'] \frac{\partial}{\partial r_j''} [\theta'' - \theta'] = \chi \frac{\partial^2 \theta''}{\partial r_j'' \partial r_j''}, \tag{4.10}$$

where we have used the fact that $\partial \theta' / \partial r_j'' = 0$. Subtracting equation (4.9) evaluated at $\boldsymbol{r}'$ from equation (4.10) yields

$$\begin{aligned} \frac{\partial}{\partial t}[\theta'' - \theta'] &+ \left(u_j(\boldsymbol{x} + \boldsymbol{r}', t) \frac{\partial}{\partial x_j} [\theta(\boldsymbol{x} + \boldsymbol{r}'', t) - \theta(\boldsymbol{x} + \boldsymbol{r}', t)] \right)_{\boldsymbol{x}=0} \\ &+ [u_j'' - u_j'] \frac{\partial}{\partial r_j''} [\theta'' - \theta'] = \chi \left[\frac{\partial^2 (\theta'' - \theta')}{\partial r_j'' \partial r_j''} + \frac{\partial^2 (\theta'' - \theta')}{\partial r_j' \partial r_j'} \right], \end{aligned} \tag{4.11}$$

where we have utilized the identity that $\partial^2 \theta' / \partial r_j'' \partial r_j'' = \partial^2 \theta'' / \partial r_j' \partial r_j' = 0$. Denote by $\boldsymbol{r} = \boldsymbol{r}'' - \boldsymbol{r}'$ the vector separating the two points $\boldsymbol{r}''$ and $\boldsymbol{r}'$. Multiplying equation (4.11) by $\theta'' - \theta'$, and denoting $\Delta_r \theta = \Delta\theta(\boldsymbol{r}', \boldsymbol{r}, t) = \theta'' - \theta'$, we find

$$\begin{aligned} \frac{\partial (\Delta_r \theta)^2}{\partial t} &+ \left(u_k(\boldsymbol{x} + \boldsymbol{r}', t) \frac{\partial}{\partial x_k} [\theta(\boldsymbol{x} + \boldsymbol{r}'', t) - \theta(\boldsymbol{x} + \boldsymbol{r}', t)]^2 \right)_{\boldsymbol{x}=0} + \frac{\partial \Delta_r u_k (\Delta_r \theta)^2}{\partial r_k''} \\ &= \chi \left[\frac{\partial^2 (\Delta_r \theta)^2}{\partial r_k'' \partial r_k''} + \frac{\partial^2 (\Delta_r \theta)^2}{\partial r_k' \partial r_k'} - 2 \left(\frac{\partial \Delta_r \theta}{\partial r_k''} \frac{\partial \Delta_r \theta}{\partial r_k''} + \frac{\partial \Delta_r \theta}{\partial r_k'} \frac{\partial \Delta_r \theta}{\partial r_k'} \right) \right], \end{aligned} \tag{4.12}$$

where we have used that

$$2f\frac{\partial^2 f}{\partial x_k \partial x_k} = \frac{\partial^2 f^2}{\partial x_k \partial x_k} - 2\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}.$$

At this point we take averages in the pure ensemble. We choose any point $\boldsymbol{x}_0$, and any diameter $R > |\boldsymbol{r}| = r$ such that the points $\boldsymbol{r}''$ and $\boldsymbol{r}'$ are contained in the ball $B(\boldsymbol{x}_0, R/2)$ which in turn is contained in G, and proceed to take averages in the ensemble of fixed $N_R(\boldsymbol{x}_0, t) = n$, where n is any positive number. Under the hypothesis that the turbulence in the pure ensemble is stationary, locally homogeneous and isotropic, the average of the first term in equation (4.12) vanishes because of stationarity and the expectation of the second term vanishes because of local homogeneity. Therefore, from equation (4.12) we obtain

$$\frac{\partial}{\partial r_k}\langle \Delta_r u_k (\Delta_r \theta)^2 | n; R\rangle = 2\chi \frac{\partial^2}{\partial r_l \partial r_l}\langle (\Delta_r\theta)^2 | n; R\rangle - 4\chi \langle \frac{\partial \Delta_r \theta}{\partial r_l'}\frac{\partial \Delta_r \theta}{\partial r_l'} | n; R\rangle, \tag{4.13}$$

where we have implied that θ is a locally homogeneous random field. Therefore,

$$\left\langle \frac{\partial^2}{\partial r_l'' \partial r_l''}(\Delta_r\theta)^2 | n; R \right\rangle = \left\langle \frac{\partial^2}{\partial r_l' \partial r_l'}(\Delta_r\theta)^2 | n; R \right\rangle = \frac{\partial^2}{\partial r_l \partial r_l}\left\langle (\Delta_r\theta)^2 | n; R \right\rangle, \tag{4.14}$$

and

$$\left\langle \frac{\partial \Delta_r \theta}{\partial r_l''}\frac{\partial \Delta_r \theta}{\partial r_l''} | n; R \right\rangle = \left\langle \frac{\partial \Delta_r \theta}{\partial r_l'}\frac{\partial \Delta_r \theta}{\partial r_l'} | n; R \right\rangle. \tag{4.15}$$

Because $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\theta(\boldsymbol{x}, t)$ are locally isotropic random fields, we have

$$\left\langle \Delta_r u_i (\Delta_r\theta)^2 | n; R \right\rangle = \left\langle \Delta_r u_r (\Delta_r\theta)^2 | n; R \right\rangle \frac{r_i}{r} \tag{4.16}$$

(recall that $\Delta_r u_r = \hat{\boldsymbol{r}} \cdot \Delta_r \boldsymbol{u}$), where $\langle \Delta_r u_r (\Delta_r\theta)^2 | n; R\rangle$ depends on $\boldsymbol{r}$ only through its absolute value r. Local isotropy also implies that $\langle (\Delta_r\theta)^2 | n; R\rangle$ is a function of the absolute value of $\boldsymbol{r}$. Therefore, in d dimensions, we have

$$\frac{\partial}{\partial r_k}\langle \Delta_r u_k (\Delta_r\theta)^2 | n; R\rangle = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\langle \Delta_r u_r (\Delta_r\theta)^2 | n; R\rangle\right), \tag{4.17}$$

and

$$\frac{\partial^2}{\partial r_l \partial r_l}\langle (\Delta_r\theta)^2 | n; R\rangle = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\langle (\Delta_r\theta)^2 | n; R\rangle\right). \tag{4.18}$$

Further, $\langle (\partial \Delta_r \theta / \partial r_l')/(\partial \Delta_r \theta / \partial r_l') | n; R\rangle = n$. Thus equation (4.13) can be rewritten as

$$\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}\left(\langle \Delta u_r (\Delta\theta)^2 | n; R\rangle - 2\chi\langle (\Delta\theta)^2 | n; R\rangle\right)\right] = -4n \tag{4.19}$$

which can be integrated easily to yield

$$\langle \Delta_r u_r (\Delta_r\theta)^2 | n; R\rangle - 2\chi\langle (\Delta_r\theta)^2 | n; R\rangle = -\frac{4}{d}nr. \tag{4.20}$$

Note that this derivation would not have changed if we had assumed that the pure ensemble was determined not only by N_R, but also by ϵ_R. In such a case, and for $d = 3$, equation (4.20) yields equation (4.2).

4.3. *Consistency between RSHP and the equations (4.5) and (4.6)*

A sensible theoretical test of RSHP is to show that they are consistent with equation (4.5). To do this, let us rewrite the first RSHP as

$$(\Delta_r\theta)^2 = V_\theta^2 \frac{rN_r}{(r\epsilon_r)^{1/3}}. \tag{4.21}$$

From the second KRSH, namely that $\Delta_r u_r = V(r\epsilon_r)^{1/3}$, we can write $\Delta_r u_r(\Delta_r\theta)^2 = (VV_\theta^2)rN_r$. Taking averages in the pure ensemble we obtain

$$\langle \Delta u_r(\Delta\theta)^2 | \epsilon_r, N_r; r\rangle = \langle VV_\theta^2 | \epsilon_r, N_r; r\rangle rN_r. \tag{4.22}$$

Now using the second RSHP, we find (for $Re_r \gg 1$ and $Pe_r \gg 1$) that $\langle VV_\theta^2|\epsilon_r, N_r; r\rangle$ is independent of ϵ_r and N_r, and is therefore a universal number. Thus, we have shown that the hypotheses are consistent with equation (4.5). Furthermore, this consistency also demands that $\langle VV_\theta^2|\epsilon_r, N_r; r\rangle = -\frac{4}{3}$. It is clear that the precise values of the different moments of V_θ do not follow from the hypotheses. The reason is simply that the second refined hypothesis postulates the existence of a universal p.d.f. for V_θ, but not its functional form.

In the limit of $Re_r \ll 1$ and $Pe_r \ll 1$, on the other hand, we can use equation (4.6) to study the dependences of $\langle V_\theta^2|\epsilon_r, N_r; r\rangle$. It can be easily checked from the definition of V_θ that equation (4.6) yields

$$\langle V_\theta^2 | \epsilon_r, N_r; r\rangle \approx \frac{Pe_r}{3} \tag{4.23}$$

indicating a dependence on Pe_r only, consistent with the RSHP.

We now turn to the experimental assessment of RSHP.

5. Experimental details

For the experimental verification of the refined hypotheses, one needs to measure simultaneously the dissipation rates of energy and scalar variance. Such measurements were attempted in the wake of a circular cylinder mounted in a subsonic wind tunnel. The cylinder had a diameter of 1.9 cm and a length of 76 cm (spanning the width of the wind tunnel), and was heated uniformly by internal heating elements. Measurements were made at a streamwise distance of 80 diameters behind the cylinder on the wake centreline. The maximum excess temperature at the measuring station was about 2.5°C, so that the heating can be considered effectively passive. The velocity of the oncoming uniform stream was 9.5 m s^{-1}. The Reynolds number based on the cylinder diameter was 12000, and the microscale Reynolds number based on the root-mean-square velocity ($u' = 40$ cm s^{-1}) and the Taylor microscale ($\lambda = 0.6$ cm) was 160. The flow is thus only of moderate Reynolds number. The limitations imposed by the moderate Reynolds number of the wake flow will be discussed later. (Unfortunately, the *joint* atmospheric velocity/temperature data, obtained about 6 m above the ground, had convergence problems and so could not be used.) On the wake centreline, the estimated Kolmogorov scale, η, was 0.023 cm, and the dissipating scale for the temperature field, $\eta_\theta = (\chi^3/\langle\epsilon\rangle)^{1/4}$, was 0.028 cm.

The measurement probe consisted of a probe support on which were mounted a hot wire (5 μm diameter, 0.5 mm long) for measuring the velocity fluctuations and a cold wire (0.6 μm diameter, 0.5 mm long) for measuring temperature fluctuations. Several detailed tests were conducted on the optimal distance between the two probes: too small a distance would result in the interference of the hot wire on the cold wire, but too large a distance would render the assumption of spatial simultaneity invalid. These tests, available in the form of two unpublished documents by Kailasnath (1988*a*, *b*), showed that the optimal separation distance was 1.2 mm for the wake; mounting the two wires closer would produce perceptible distortion of the temperature signal. By increasing the distance even further and studying the effect on the joint statistics of the measured velocity and temperature signals, it was determined that this distance, although larger than is ideal, was not unacceptably large. The hot wire had a flat frequency response up to 40 kHz, the corresponding number for the cold wire was 4 kHz. The hot wire was operated on a DANTEC constant-temperature anemometer typically at an overheat of 1.3; larger overheat, while desirable for better signal/noise ratio, would produce unacceptable cross-talk between the signals from the two wires. The signal-to-noise ratio for these operating conditions was estimated to be 55 dB. The cold wire was operated on a constant-current anemometer built in-house on the basis of a design by Peattie (1987); the operating current of 120 μA produced a temperature signal/noise ratio of 40 dB. The signals from the hot and cold wires were sampled at a frequency of 8 kHz. Thirty two data files, each consisting of 1.2×10^5 data points were obtained.

From these velocity and temperature time traces, derivatives were obtained by digitally differentiating the signals. The time derivatives were in turn treated as space derivatives in the direction of the mean motion of the fluid by invoking Taylor's frozen flow hypothesis. Much literature exists on the validity or otherwise of Taylor's hypothesis (see, for example, Antonia, Chambers & Phan-Thien 1980 and the papers cited there), but its use is generally accepted as reasonable for small scales (such as the derivative quantities) if the turbulence level is small (see, e.g., Monin & Yaglom 1971). In this instance, the turbulence level (as estimated by the ratio of the measured root-mean-square streamwise velocity fluctuation to the mean velocity) was 4.2%, and may be considered small enough.

6. Experimental results

6.1. *The extent of the inertial range*

Since the Reynolds number of the wake is modest, the extent of the inertial range – if one exists – should be examined explicitly. This can be done by testing the degree to which the well-known relation due to Kolmogorov (1941*b*), exactly valid in a limiting sense in the inertial range, is satisfied by the wake measurements. This relation is

$$\langle (\Delta_r u_r)^3 \rangle = -\tfrac{4}{5}\langle \epsilon \rangle r. \tag{6.1}$$

For notational convenience we will drop the primes in V_θ', ϵ' and N' in this section (including figures). The use of primes will be resumed in §7, where we present our conclusions.

Figure 1 shows a plot of $\langle (\Delta_r u_r)^3 \rangle / (\langle \epsilon \rangle r)$ versus r/η. An optimistic guess is that the inertial range is about half a decade, roughly as marked in the figure. Note that the plateau in figure 1 is somewhat less than the 4/5 required by equation (6.1). This is not an unusual feature of several other measurements in the literature (e.g. Anselmet *et al.*, figure 10), and could have a variety of causes such as the moderate value of the Reynolds number, use of discrete derivatives in the estimation of $\langle \epsilon \rangle$, use of the Taylor hypothesis in treating time increments as spatial increments, and, finally, the use of ϵ' instead of ϵ.

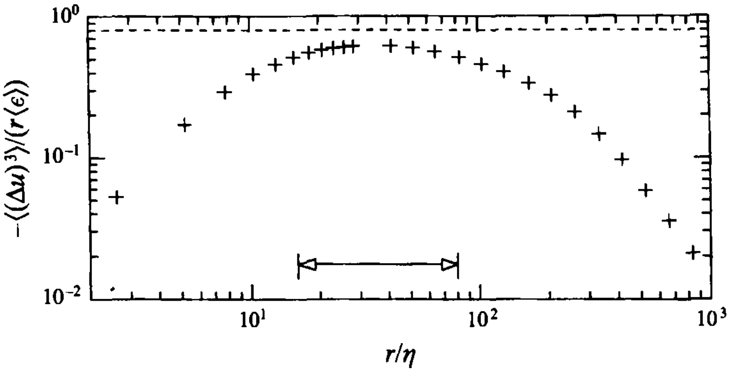

FIGURE 1. The third-order structure function, normalized with $r\langle\epsilon\rangle$, as a function of the separation distance r in units of the Kolmogorov scale η. The extent of the plateau interpreted roughly as the inertial range is marked; this is taken as the range in which the variation of the structure function is within about ±5% near the peak region.

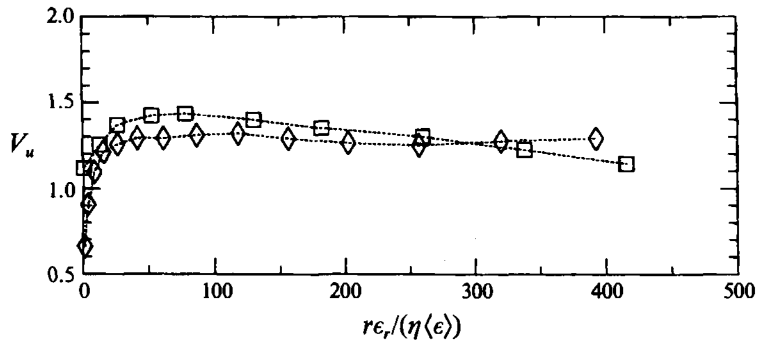

FIGURE 2. The standard deviation of the stochastic variable $V = \Delta_r u_r/(r\epsilon_r)^{1/3}$ plotted as a function of $r\epsilon_r$ normalized with $\eta\langle\epsilon\rangle$, for the atmospheric boundary layer (diamonds, $r/\eta = 144$) and for the wake of a cylinder (squares, $r/\eta = 78$). The former is well within the inertial range, and the latter marks the upper end of the inertial range. The mean value of V is close to zero in both cases.

Given the unsatisfactory extent of the inertial range, it was felt desirable to ascertain that the quantities computed from the moderate-Reynolds-number data indeed correspond meaningfully to the inertial range. We had acquired in a previous study (Stolovitzky *et al.* 1992) large amounts of velocity data in the atmospheric surface layer at high Reynolds number (microscale Reynolds number of about 2000), and felt that detailed comparisons of the velocity statistics between the wake velocity data and the atmospheric *velocity* data (which *do* converge) would throw some light on the present issue. Figure 2 shows one such example. It compares the standard deviation of the universal variable V plotted against $r\epsilon_r/(\eta\langle\epsilon\rangle)$; the data correspond to the atmospheric surface layer data (diamonds) for the separation distance $r/\eta = 144$ which is within the inertial range, as well as to the wake data (squares) at the upper end of the presumed inertial range marked in figure 1. The wake data blend reasonably well with the atmospheric data, demonstrating the reasonableness of assuming that there is an inertial range – albeit a small one – in the wake data.

Similar tests for the temperature data could not be made because of convergence problems of the atmospheric temperature data already mentioned. There is indeed no direct proof that the small structure of the scalar has attained a universal state, but it is expected that the data to be presented below will shed some light also on this issue. For this reason, as well as for other limitations of the present data as mentioned in §5, an independent effort at high Reynolds numbers will be both welcome and timely.

6.2. *Experimental test of RSHP*

To test RSHP, we proceed as follows. For given windows of ϵ_r and N_r, we compute the normalized histograms of $V_\theta = \Delta_r\theta(r\epsilon_r)^{1/6}/(rN_r)^{1/2}$. According to the hypotheses, we expect that V_θ will depend on both ϵ_r and r for small values of r. If r is large enough (still much smaller than L), so is the local Reynolds number (recalling that $Re_r = r^{4/3}\epsilon_r^{1/3}/\nu$). (Note that because $Pr = 0.7 \sim O(1)$ for our experimental conditions, taking $Re_r \gg 1$ also implies $Pe_r = Re_r Pr \gg 1$.) For this case, according to the second hypothesis, the p.d.f.s are expected to become independent of ϵ_r and r.

Figure 3 shows plots of the p.d.f. of V_θ for $r/\eta = 78$ which, according to figure 1, corresponds roughly to the upper end of the inertial range. Each of figure 3(*a*)–3(*d*) corresponds to different windows of N_r, as indicated; within each, different p.d.f.s correspond to different windows of ϵ_r specified in the caption. In spite of the significant scatter, it appears reasonable to say that $P_T(V_\theta|\epsilon_r, N_r; r)$, the conditional p.d.f. of V_θ for given ϵ_r and N_r, shows a modest collapse onto a unique p.d.f., which is Gaussian-like in shape. The existence of such a unique p.d.f. which is independent of r, ϵ_r and N_r in the inertial range, for which the local Reynolds number is sufficiently high, is the substance of RSHP.

The predictive power of the hypothesis can be tested by analysing experimental data further. It would have been ideal to test equation (4.5) directly from the data, but the convergence of the third-order moment is poor in the experimental data. So we shall test an alternative feature of the refined hypotheses, namely, that for large enough values of the local Reynolds numbers, the conditional expectation of $|\Delta_r u_r|(\Delta_r\theta)^2$ in the pure ensembles is proportional to rN_r and independent of ϵ_r. That is,

$$\langle|\Delta_r u_r|(\Delta_r\theta)^2|\epsilon_r, N_r; r\rangle = KrN_r \tag{6.2}$$

where K is a constant. Thus, $\langle|\Delta_r u_r|(\Delta_r\theta)^2|\epsilon_r, N_r; r\rangle$ should be independent of ϵ_r for all values of rN_r on which it depends linearly. Figure 4 shows the conditional expectation $\langle|\Delta_r u_r|(\Delta_r\theta)^2|\epsilon_r, N_r; r\rangle$ as functions of both ϵ_r and rN_r. It is seen from figure 4(*a*) that $\langle|\Delta_r u_r|(\Delta_r\theta)^2|\epsilon_r, N_r; r\rangle$ depends linearly on rN_r, as expected. Figure 4(*b*) shows that the dependence on ϵ_r is quite weak, also as expected. These results provide additional confirmation of RSHP.

For r in the dissipation region, the conditional p.d.f.s depend on both ϵ_r and rN_r. These data are not shown in detail here. A quick indication of this is provided by figures 5(*a*) and 5(*b*) for $r/\eta = 13$, which lies in the dissipation range. Not surprisingly, equation (6.2) is not verified: in this case we have the postulated proportionality of the moment on rN_r, but there exists an additional dependence on ϵ_r. This dependence is well fitted by a power law $\epsilon_r^{1/2}$ (figure 5*b*).

The dependence seen in figure 5(*b*) is related to the modest value of the local Reynolds number corresponding to $r/\eta = 13$. To explain this dependence, note that, for small values of r, a Taylor expansion of the left-hand side of equation (6.2) yields

$$\langle|\Delta_r u_r|(\Delta_r\theta)^2|\epsilon_r, N_r; r\rangle = S\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2|\epsilon_r, N_r; r\right\rangle^{1/2} \times \left\langle\left(\frac{\partial \theta}{\partial x_1}\right)^2|\epsilon_r, N_r; r\right\rangle r^3, \tag{6.3}$$

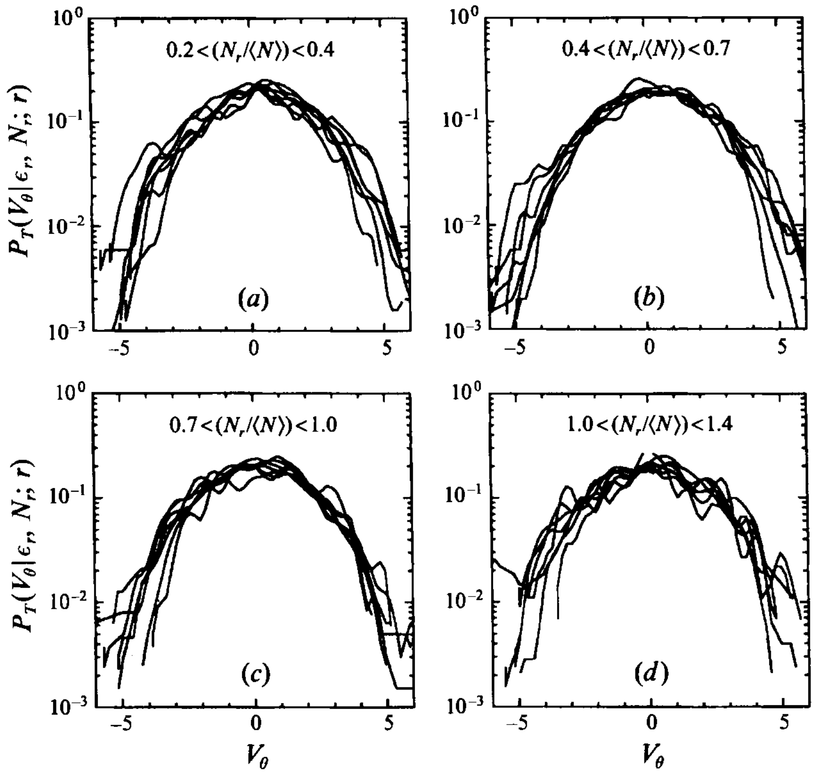

FIGURE 3. Conditional p.d.f.s of $V_\theta = \Delta_r\theta(r\epsilon_r)^{1/6}/(rN_r)^{1/2}$ given ϵ_r and N_r, for $r/\eta = 78$. Each part corresponds to the indicated windows of $N_r/\langle N\rangle$; for the curves in each part, the windows of $\epsilon_r'/\langle\epsilon\rangle$ are: (0.4,0.6), (0.6,0.8), (0.8,1.0), (1.0,1.4), (1.4,1.8), (1.8,2.4), (2.4,3.2). The minimum number of points utilized to construct the p.d.f.s corresponded (for all windows of $N_r/\langle N\rangle$) to the last window of $\epsilon_r/\langle\epsilon\rangle$ and was approximately 14000 in (*a*), 9000 in (*b*), 7000 in (*c*), and 9000 in (*d*). The large scatter in the p.d.f.s, in spite of the large data base used here, is due to the relatively small number of samples resulting from the conditioning that needs to be done on both ϵ_r and N_r. The conditional p.d.f.s look the same for several other values of r in the inertial range, and so one can improve their convergence by averaging over these values of r/η in the inertial range, but this has not been attempted.

where

$$S = \frac{\left\langle \left|\frac{\partial u_1}{\partial x_1}\right| \left(\frac{\partial \theta}{\partial x_1}\right)^2 | \epsilon_r, N_r; r \right\rangle}{\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 | \epsilon_r, N_r; r \right\rangle^{1/2} \left\langle \left(\frac{\partial \theta}{\partial x_1}\right)^2 | \epsilon_r, N_r; r \right\rangle}. \tag{6.4}$$

For a pure ensemble, $\langle (\partial u_1/\partial x_1)^2 | \epsilon_r, N_r; r\rangle = \epsilon_r/15\nu$ (see Stolovitzky 1994), and $\langle (\partial\theta/\partial x_1)^2 | \epsilon_r, N_r; r\rangle = N_r/3\chi$. Therefore, for small r, we obtain the result that

$$\langle |\Delta_r u_r| \Delta_r\theta^2 | \epsilon_r, N_r; r\rangle \approx \frac{S}{3\sqrt{15 Pr}} \frac{r^2 \epsilon_r^{1/2}}{\nu^{3/2}} r N_r. \tag{6.5}$$

Figure 5, in combination with equation (6.5), suggests that S for small r is independent of ϵ_r and N_r or depends on them only weakly.

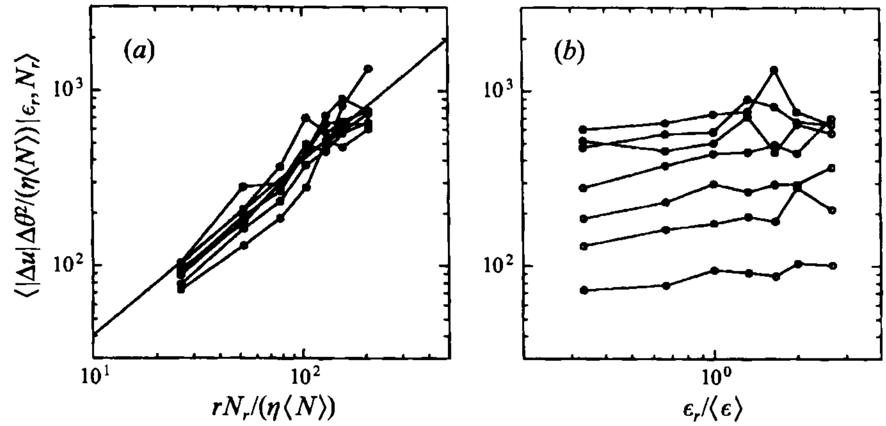

FIGURE 4. The conditional expectation in the inertial range of the product of the absolute value of the velocity difference times the square of the temperature difference, given ϵ_r and N_r. The conditioning values of $\epsilon_r/\langle\epsilon\rangle$ and $N_r/\langle N\rangle$ were (0.2, 0.4), (0.4, 0.7), (0.7, 1.0), (1.0, 1.4), (1.4, 1.7), (1.7, 2.0), (2.0, 2.7). The temperature and velocity increments were taken across the separation distance $r/\eta = 78$. In (*a*), the data lie parallel to the straight solid line of slope 1, as expected by RSHP. In (*b*), the data show no dependence on ϵ_r, also in accordance with RSHP.

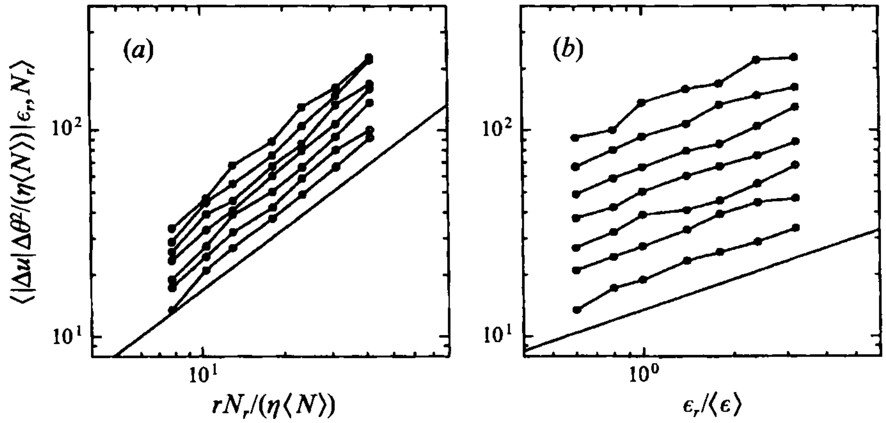

FIGURE 5. The temperature and velocity increments were taken across the separation distance $r/\eta = 13$. In (*a*), the data are approximately parallel to the solid line of slope 1, and there is a systematic dependence on ϵ_r, as expected by RSHP. This dependence is studied in (*b*), which shows that the data lie parallel to the straight line of slope 1/2, for which an explanation is provided in the text. See (6.5).

7. Summary and conclusions

In this paper, we have stated for passive scalar fields mixed by high-Reynolds-number turbulence the formal equivalent of Kolmogorov's refined similarity hypotheses for the velocity field. We have shown that the refined similarity hypotheses for passive scalars (RSHP) are consistent with two theoretical results obtained from the evolution equation for passive scalars. We have also shown that experimental data obtained in a moderate-Reynolds-number turbulent wake support RSHP.

The theoretical result is obtained from the assumption of local isotropy for Obukhov's pure ensembles. This is a more restrictive assumption than that of conventional local isotropy for the global domain; the result obtained, namely equation (4.4) (which in the appropriate limits yields equations (4.5) and (4.6)), is also stronger than the corresponding ones due to Yaglom for global averages. From the present theoretical results, we have shown that the conditional expectation $\langle VV_\theta^2 | \epsilon_r, N_r; r\rangle$ is independent of ϵ_r and N_r, as demanded by RSHP.

Experimentally, we have obtained the p.d.f.s of V'_θ conditioned on ϵ'_r and N'_r, and shown that they are essentially independent of r, ϵ'_r and N'_r in the inertial range. Also verified is the linear dependence of the conditional expectation $\langle |\Delta_r u_r| (\Delta_r \theta)^2 | \epsilon'_r, N'_r; r\rangle$ on rN'_r and its independence of ϵ'_r, consistent with RSHP. For smaller r, the latter result is not obeyed by the data (as expected from RSHP); we have examined this dependence and offered a simple explanation on the basis of Taylor series expansions valid for small enough r. While it is true that experiments at higher Reynolds numbers would have been desirable, it is our belief that RSHP can be considered to have been verified approximately in these experiments.

Finally, it is curious to note that the passive scalar fields, which exhibit various deviations from local isotropy at moderate Reynolds numbers, obey RSHP reasonably well. This would suggest that RSHP are not sensitive tests of universality. It is conceivable that departures from universality are hidden by the process of taking conditional expectations. This was our conclusion from another study (Stolovitzky & Sreenivasan 1994) where we showed that stochastic processes other than turbulence show most features of the refined similarity hypotheses for the velocity.

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REFERENCES

Anselmet, F., Gagne, Y., Hopfinger, E. J. & Antonia, R. A. 1984 *J. Fluid Mech.* **140**, 63.

Antonia, R. A., Chambers, A. J. & Phan-Thien, N. 1980 *Boundary-Layer Met.* **19**, 19.

Antonia, R. A., Hopfinger, E. J., Gagne, Y. & Anselmet, F. 1984 *Phys. Rev.* A **30**, 2704.

Antonia, R. A. & Van Atta, C. W. 1975 *J. Fluid Mech.* **67**, 273.

Batchelor, G. K. 1959 *J. Fluid Mech.* **5**, 113.

Batchelor, G. K., Howells, I. D & Townsend, A. A. 1959 *J. Fluid Mech.* **5**, 134.

Batchelor, G. K. & Townsend, A. A. 1948 *Proc. R. Soc. Lond.* A **199**, 238.

Chen, S., Doolen, G. D., Kraichnan, R. H. & She, Z.-S. 1993 *Phys. Fluids* A **5**, 458.

Chen, S., Doolen, G. D., Kraichnan, R. H. & Wang, L.-P. 1995 *Phys. Rev. Lett.* **74**, 1755.

Corrsin, S. 1951 *J. Appl. Phys.* **22**, 469.

Hosokawa, I. 1993 *J. Phys. Soc. Japan* **62**, 10.

Hosokawa, I. 1994 *Phys. Rev.* E **49**, R4775.

Kailasnath, P. 1988*a* Simultaneous measurements of the velocity and temperature in a heated turbulent wake. Technical Report, Mason Laboratory, Yale University.

Kailasnath, P. 1988*b* Correlation between the dissipation fields of temperature and velocity in turbulent flows. Technical Report, Mason Laboratory, Yale University.

Kolmogorov, A. N. 1941*a* *Dokl. Akad. Nauk SSSR* **30**, 301.

Kolmogorov, A. N. 1941*b* *Dokl. Akad. Nauk SSSR* **32**, 19.

Kolmogorov, A. N. 1962 *J. Fluid Mech.* **13**, 82

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics.* Pergamon Press.

Meneveau, C. & Sreenivasan, K. R. 1991 *J. Fluid Mech.* **224**, 429.

Meneveau, C., Sreenivasan, K. R., Kailasnath, P. & Fan, M.S. 1990 *Phys. Rev. A* **41**, 894.

Monin, A. S. & Yaglom, A. M. 1971 *Statistical Fluid Mechanics*, Vol. I. MIT Press.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics*, Vol. II. MIT Press.

OBUKHOV, A. M. 1949 *Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz.* **13**, 58.
OBUKHOV, A. M. 1962 *J. Fluid Mech.* **13**, 77.
PEATTIE, R. 1987 *J. Phys.* E **20**, 565.
PRASKOVSKY, A. A. 1992 *Phys. Fluids* A **4**, 2589.
SREENIVASAN, K. R. 1991 *Proc. R. Soc. Lond.* A **434**, 165.
SREENIVASAN, K. R., ANTONIA, R. A. & DANH, H. Q. 1977 *Phys. Fluids* **20**, 1238.
SREENIVASAN, K. R. & MENEVEAU, C. 1988 *Phys. Rev.* A **38**, 6287.
STOLOVITZKY, G. 1994 The statistical order of small scales in turbulence. PhD thesis, Yale University, New Haven, Connecticut.
STOLOVITZKY, G., KAILASNATH. P. & SREENIVASAN, K. R. 1992 *Phys. Rev. Lett.* **69**, 1178.
STOLOVITZKY, G. & SREENIVASAN, K. R. 1994 *Rev. Mod. Phys.* **66**, 229.
THORODDSEN, S. T. 1995 *Phys. Fluids* A **7**, 691.
THORODDSEN, S. T. & VAN ATTA, C. W. 1992*a* *Phys. Fluids* A **4**, 2592.
THORODDSEN, S. T. & VAN ATTA, C. W. 1992*b* *J. Fluid Mech* **244**, 547.
VAN ATTA, C. W. 1971 *Phys. Fluids* **14**, 1803.
WANG, L.-P., CHEN, S., BRASSEUR, J. G. & WYNGAARD, J. C. 1994 Examination of fundamental hypotheses in the Kolmogorov refined turbulence theory through high-resolution simulation. Part 1. Velocity field. *J. Fluid Mech.* (submitted).
YAGLOM, A. M. 1949 *Dokl. Akad. Nauk.* **69**, 743.